

A Covariant Geometrical Representation of Quantum Interacting Electrons

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A study of fundamental geometrical interactions shows that the Dirac electron can be represented as a conformal wave. A Riemannian space is used, having coordinates that transform locally as spinors. The wave function becomes a gradient. Anti-commuting matrices map the eight-dimensional waves into five dimensions. A projection into space time gives the known gravitational and electromagnetic forces. The electron transforms into a neutrino under hyper-rotations. First quantization is automatic. The theory is covariant and contains the known electron interactions.

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INTRODUCTION

The objective of this article is to introduce a fundamental geometrical description of the relativistic electron. Following the initial development by Dirac [1, 2], and subsequent studies by mathematicians and physicists [3], the understanding of fundamental spin has continued to be a major enigma of modern physics. The opportunity to describe the electron as a Riemannian structure in a curvilinear space is made possible by the development of a covariant five dimensional theory [4] that describes the interactions of quantum particles. A recent discussion [5] gives a general overview and some mathematical preliminaries. The eight-dimensional formalism predicts the known properties of electrons when projected through five dimensions onto space-time.

The present work initiates the development of a class of exact equations which describe the quantum properties of spin-1/2 particles. The usual representation of point interactions of elementary particles is to be replaced by geometrical transformations that follow the particle field continuously through the interaction. To do this, an individual Riemannian structure is assigned to each particle. No classical-like point objects are used.

Fields in five or eight dimensions are represented in a single universal coordinate system. From the curvature tensors, a first quantized structure is generated without active quantization [6], and a quantum field is developed without reference to a classical basis. This geometrical paradigm insures compatibility with general relativity. Mass appears during the reduction from five to four dimensions. It satisfies gravitation equivalence and has mechanical inertia that is appropriate to quantum particles.

An absolute equivalence for all interactions is assumed. Internal transformations convert between different types of force. Each particle has a full set of fields, including a metric, vector potential, and a wave function. These describe motion and interaction. The equivalence transformations are analogous with, and in some cases identical to, the gauge transformations used in quantum field theory. The wave functions, and also the interaction mechanisms, are associated with conformal factors that

appear naturally. Characteristic linear wave equations come from the invariant conformal waves.

Fundamental interactions are all taken to be time-symmetric. Sufficient distant absorbing particles are assigned to account for the retarded behavior of electromagnetism. The resulting structure is a set of mathematically determined, non-linear, interacting wave fields. Second quantization is approached as a formalism to account for multiple particles. The combined set of fields evolves without any assumption of fundamental statistics. Photons and gravitons acquire their discreet properties entirely from the countability of the emitting particles. No free fields are used.

FIVE-DIMENSIONAL THEORY

Because some of the properties of five dimensional theory are required, a short summary is given here. The five coordinates, x^m , $m = 0, \dots, 4$, can, in a local Riemannian system, be taken equal to (t, x, y, z, τ) . The usual four-metric is

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (1)$$

and becomes [10]

$$0 = g_{\mu\nu} dx^\mu dx^\nu - (-\mathcal{A}_\mu dx^\mu - d\tau)^2 \equiv \gamma_{mn} dx^m dx^n \quad (2)$$

which gives

$$\gamma_{mn} \equiv \begin{pmatrix} g_{\mu\nu} - \mathcal{A}_\mu \mathcal{A}_\nu & -\mathcal{A}_\mu \\ -\mathcal{A}_\nu & -1 \end{pmatrix} \quad (3)$$

where, in this gauge,

$$\mathcal{A}_\nu = \frac{1}{m} \left[\frac{\partial}{\partial x^\nu} \Im(\ln \psi) - e A_\nu \right] \quad (4)$$

All curvature is taken to be conformally generated, so that the Ricci tensor of $\omega \gamma^{mn}$ for some function ω is identically zero. This implies that source terms exist and that they can be calculated from ω . A particular choice gives

the gravitational and electrodynamic field equations. In four-dimensional form these are

$$R^{\alpha\beta} = 8\pi\kappa \left(F^\alpha{}_\mu F^{\mu\beta} + m|\psi|^2 \mathcal{A}^\alpha \mathcal{A}^\beta + m|\psi|^2 \frac{1 - \mathcal{A}^2}{2 - \mathcal{A}^2} g^{\alpha\beta} \right) \quad (5)$$

$$F^{\beta\mu}|_\mu = 4\pi e|\psi|^2 \mathcal{A}^\beta. \quad (6)$$

The quantum field equation can be found by working from the invariant curvature scalar. Setting it to zero gives

$$\frac{1}{\sqrt{-g}} \left(i\hbar \frac{\partial}{\partial x^\mu} - eA_\mu \right) \sqrt{-g} g^{\mu\nu} \left(i\hbar \frac{\partial}{\partial x^\nu} - eA_\nu \right) \psi = \left[m^2 + \frac{3}{16} \left(\dot{R} - \frac{e^2}{4m^2} F_{\alpha\beta} F^{\alpha\beta} \right) \right] \psi. \quad (7)$$

Here, the fifth dimension has been reduced by setting derivatives of the conformal factor with respect to the proper time equal to the mass. This reduction insures that identical particles have equal rest mass.

EIGHT-DIMENSIONAL DERIVATION

The Dirac equation must first be written in five-dimensional form. The fifth anti-commuting spin matrix is assigned to the fifth coordinate. The commutation properties are

$$\gamma^{mn} \delta_A^B = \frac{1}{2} \{ \gamma^m, \gamma^n \} \equiv \frac{1}{2} (\gamma^m{}_A{}^C \gamma^n{}_C{}^B + \gamma^n{}_A{}^C \gamma^m{}_C{}^B) \quad \text{for } m, n = 0, \dots, 4, \text{ and } A, B, C = 1, \dots, 4. \quad (8)$$

The five-dimensional form of the Dirac equation is used [5],

$$\gamma^m{}_A{}^B \frac{\partial}{\partial x^m} \Psi_B = 0 \quad \text{or} \quad \gamma^m \frac{\partial}{\partial x^m} \Psi = 0. \quad (9)$$

To relate this to properties of conformal transformations in eight space, we choose eight real coordinates ξ_r^A, ξ_i^A for $A = 1 \dots 4$, and consider the associated conformally flat space. Let the conformal factor be denoted by ω . The condition that the scalar curvature is zero constitutes a second-order differential equation for ω and provides a Riemannian characterization of the conformal waves. The equation can be written in modern spinor notation. The eight coordinates must be arranged in complex pairs.

$$\xi^A = \xi_r^A + i\xi_i^A, \xi^{\bar{A}} = \xi_r^A - i\xi_i^A, \quad A = 1 \dots 4 \quad (10)$$

where the bar of conjugation is applied to the index for consistency with existing conventions [7]. The fundamental form is $d\xi^A d\xi^{\bar{B}} \epsilon_{A\bar{B}}$ with spinor metric

$$\epsilon_{A\bar{B}} = \epsilon^{A\bar{B}} = \text{diag}(1, 1, -1, -1). \quad (11)$$

The curvature equation is

$$\epsilon^{\bar{A}B} \frac{\partial}{\partial \xi^{\bar{A}}} \frac{\partial}{\partial \xi^B} \Psi = 0 \quad (12)$$

where $\omega = \Psi^p$ with $p = 4/(n-2) = 2/3$.

Locally, the coordinates in the neighborhood of a point in five-space are related to a corresponding point in eight-space by the differential relation

$$dx^m = \zeta^A \gamma^m{}_A{}^B d\xi^{\bar{C}} \epsilon_{\bar{C}B} + d\xi^A \bar{\gamma}^m{}_A{}^B \zeta^{\bar{C}} \epsilon_{\bar{C}B} \quad (13)$$

Because the eight-dimensional displacements are summed with conjugates, the five space displacements are always real. Furthermore Lorentz rotations, extended to five dimensions, are mapped in the usual way from the spinor space of displacements $d\xi^A, d\xi^{\bar{A}}$ to dx^m . It is the simplest transformation law which satisfies these conditions. The parameter ζ specifies the relative orientation of the spin space. It is equivalent to the use of a four- or five-dimensional spin frame.

The quantity

$$\Psi_B = \frac{\partial \Psi}{\partial \xi^B} \quad (14)$$

is taken as the Dirac spinor wave function which, following equation (12), satisfies the first order equation

$$\epsilon^{\bar{A}B} \frac{\partial \Psi_B}{\partial \xi^{\bar{A}}} = 0 \quad (15)$$

Using the chain rule

$$\frac{\partial}{\partial \xi^{\bar{A}}} = \frac{\partial x^m}{\partial \xi^{\bar{A}}} \frac{\partial}{\partial x^m}$$

from the coordinate transformation (13), this becomes

$$\epsilon^{\bar{A}B} \gamma^m{}_D{}^C \zeta^D \epsilon_{\bar{A}C} \frac{\partial \Psi_B}{\partial x^m} \equiv \zeta^D \left(\gamma^m{}_D{}^B \frac{\partial \Psi_B}{\partial x^m} \right) = 0 \quad (16)$$

The complex conjugates $\xi^{\bar{A}}$ are treated as independent of the ξ^A 's during differentiation. The quantity in parenthesis is identified with equation (9), and is interpreted as characterizing the eight-dimensional space in an orientation independent way. Equation (16) should be satisfied for any value of ζ .

Using equation (14), a local plane wave solution of (7) can be converted to a solution of (9) by differentiation.

$$\Psi = e^{i(\omega t - \vec{k} \cdot \vec{x} - m\tau)} \equiv e^{ik_m x^m}, \quad k_m = (\omega, \vec{k}, m) \quad (17)$$

$$\Psi_A \equiv \frac{\partial \Psi}{\partial \xi^A} = \Psi i k_m \frac{\partial x^m}{\partial \xi^A} = i \Psi (k_m \gamma^{\dagger m}) \zeta^\dagger \quad (18)$$

This is the form for a free electron or positron with arbitrary spin when written in the basis implied by the

matrices of equation (9). In this way, it is assured that the particle is locally a Dirac electron.

This derivation involves a minimum of physical assumptions and has no explicitly performed first quantization. Gravitational and electromagnetic interactions are included when the $\gamma^m_A{}^B$ matrices are defined. Moreover, the conformal variations that generate external source terms in the five-dimensional theory imply, through (13), that they are equivalent to analogous conformal transformations in the eight-dimensional space. Forces other than electromagnetism or gravitation must be present because the matrix of second order coordinate derivatives has become larger. Weak interactions are expected because they are known to be described with the Dirac theory, and equivalence requires that such additional forces be gauged to the others.

In local Cartesian coordinates, and using standard notation for the γ 's, the six quantities

$$A^\mu = \psi^\dagger \gamma^\mu \psi, \text{ for } \mu = 0, 1, 2, 3, \\ \text{with } A^4 = i\psi^\dagger \gamma^5 \psi, \text{ and } A^5 = i\psi^\dagger \psi \quad (19)$$

following [8, 9] combine into a quadratic invariant which, in modern notation, is

$$(A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2 = (A^4)^2 + (A^5)^2. \quad (20)$$

Term by term, in the classical limit, it corresponds to the relation $E^2 - p^2 = m^2$. Two of the six quantities must make up the mass. If these two quantities cancel, a zero-mass particle results. The condition is

$$0 = (A^4)^2 + (A^5)^2 \equiv (A^5 - iA^4)(A^5 + iA^4) \\ \equiv [\psi^\dagger(1 + \gamma^5)\psi] \cdot [\psi^\dagger(1 - \gamma^5)\psi] \quad (21)$$

One of the factors must be zero, giving either a neutrino or an anti-neutrino. Equation (15) has a zero-mass solution, and it must transform into the electron solution under a hyper-rotation. This inter-transformation is identified with weak isospin. Because (7) and (9) can only apply to a system of particles of fixed charge-to-mass ratio, neutrinos and anti-neutrinos must have, respectively, the same charge-to-mass ratio as the electron and positron. A more detailed representation of the weak force as a conformal effect will be discussed at a later time.

INTERPRETATION

Some of the properties of the postulated eight-dimensional physical space are not yet understood. The dimensional reduction does not follow the common Kaluza-Klein paradigm since spin properties persist from the base space. The success of the reduction depends on starting with a sufficiently simple structure in eight dimensions. If both the five- and eight-dimensional spaces are conformally flat, integrability issues are minimized.

The assumption of conformal flatness appears to be rich enough to describe gravitational, electromagnetic, and weak interactions, but greater complexity may be possible. It is not known what additional phenomenological predictions can be expected.

The proposed continuing methodology is to write strong interactions in eight-dimensional form, assume a universal geometry of isospin, and then try to understand what additional representations are needed. Particle transmutation by isospin hyper-rotation is distinct from the compositional change seen in bound systems, and implies a general theory of mass. Historically, each increment in the number of dimensions has demonstrated an increasingly richer collection of physical observations. The increase from five to eight dimensions may be enough to include the quarks or the gluonic interactions. More sophisticated internal representations of particle mass are expected, especially if hadrons or quarks are involved. Specific values may appear with particle properties to match.

The additive terms to the square of the electron mass, as seen in (7), are characteristic of higher-dimensional projected theories. The coefficient depends on the number of dimensions. The terms cannot be eliminated if, in a specific dimensionality, all coordinates are formally equivalent. This implies a propagational mass that is distinct from the rest mass. In a strong gravitational or electromagnetic field, the effective mass that appears in the local equation of motion is different from that in the absence of the interacting field. This effect is unobservably small for the electron, but may be measurable, eventually, for neutrinos. Possible contributions from the flavor interactions are not understood, and may be important. Also, even though strong interactions are not usually associated with electrons, they may contribute as well.

Fiber bundles, originally developed to describe spin in a curvilinear setting, continue to be relevant. An implication of this construction is that a bundle may be symptomatic of the residual effects of dimensional reduction. Some Yang-Mills theories may fall into this class. Both mathematical and physical issues are involved. It is usually claimed that the graviton, because it is spin two, cannot be the Higgs particle. Here the graviton is grouped with the photon to make a five-scalar. The usual condition may be overly restrictive. The Higgs mechanism should connect with the mass mechanisms of quantum theory and general relativity. A curvilinear methodology is required, and may naturally bring the ideas together.

CONCLUSION

The Dirac electron theory is reformulated in an eight-dimensional coordinate space. The eight parameters combine into four complex pairs that transform lo-

cally according to spinor representations of the extended Lorentz group. Anti-commuting Dirac matrices are used to map local complex displacements onto five-dimensional space. Each particle has a separate set of interacting gravitational and electromagnetic fields. The wave function is represented by the eight-gradient of a conformal parameter. If this parameter satisfies the condition of zero scalar curvature, the Dirac equation appears from the geometry. Neutrino solutions satisfy the same curvature condition and transform into electrons under hyper-rotation. This gives the isospin-changing weak interactions an explicit mechanical model. The eight dimensional formalism may be suitable for strong interactions. This construction augments the known explicit geometrical descriptions of quantum electrons moving in gravitational and electromagnetic fields to include the weak forces. Until such time as strong interactions might be observed, this provides a complete description of the electron.

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